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On a class of non-linear elastic materials

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Abstract

This article deals with a family of non-linear hyperelastic materials $\mathscr{M}(\varepsilon)$ depending on a parameter ε varying from 0 to 1; $\mathscr{M}(0)$ is a masonry-like material and $\mathscr{M}(1)$ is linear elastic. Some properties of the function $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ delivering the Cauchy stress corresponding to the infinitesimal strain $\mathbf{E}$, are proved; in particular, it is shown that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is strongly monotone for $\varepsilon > 0$ and monotone for $\varepsilon = 0$. Moreover, denoting by $[\mathbf{u}(\cdot;\varepsilon),\ \mathbf{E}(\cdot;\varepsilon),\ \mathbf{T}(\cdot;\varepsilon)]$ the solution to the equilibrium problem for solids made of a material $\mathscr{M}(\varepsilon)$, the convergence of $[\mathbf{u}(\cdot;\varepsilon),\ \mathbf{E}(\cdot;\varepsilon),\ \mathbf{T}(\cdot;\varepsilon)]$ for ε going to 0 and 1, is investigated.

Keywords: Non-linear elasticity; Masonry-like materials; Equilibrium problem

1. Introduction

This article deals with a family of non-linear elastic materials $\mathscr{M}(\varepsilon)$ dependent on parameter ε varying from 0 to 1 such that $\mathscr{M}(1)$ is linear elastic and $\mathscr{M}(0)$ corresponds to the masonry-like material described in Del Piero (1989) and Lucchesi et al. (1994) . Specifically, for a fixed ε, I introduce a partition of the strain space in order to define the function $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ which gives the Cauchy stress corresponding to the infinitesimal strain $\mathbf{E}$. $\mathscr{M}(\varepsilon)$ materials are a generalization of the conewise linear elastic materials introduced in Curnier et al. (1995). In fact, while in Curnier et al. (1995), the strain space is divided into convex polyhedral cones and the stress is assumed to be linear in each cone, here instead the partition elements are not necessarily convex, and the stress $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is a non-linear function of $\mathbf{E}$. As in Curnier et al. (1995), it is proved that continuity of the stress–strain law is the key property for globalizing a piecewise property. In particular, I prove that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is monotone with respect to $\mathbf{E}$ in each of the domains $\mathscr{R}_i(\varepsilon)$, $i = 1, 2, 3$, into which the strain space is divided. This result, together with the continuity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ at the interfaces of domains $\mathscr{R}_i(\varepsilon)$ guarantees that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is globally monotone.

The materials introduced in this article are a generalization of the bimodular materials described in Green and Mkrtichian (1977) and Jones (1977), which exhibit different behavior when stressed through compression rather than tension. If Poisson's ratio equals zero, the two materials coincide.

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In Section 2, the materials $\mathscr{M}(\varepsilon)$ are described, and the properties of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ examined. In particular, the strong monotonicity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ for $\varepsilon > 0$ and the Lipschitz continuity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ for every ε are proved. Moreover, the derivative $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ with respect to $\mathbf{E}$ is calculated in each domain $\mathscr{R}_i(\varepsilon)$, and proved to be positive definite for $\varepsilon > 0$ and positive semi-definite for $\varepsilon = 0$, by explicitly calculating the eigenvalues of $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$. Finally, the dependence of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ on ε for a fixed $\mathbf{E}$ is analyzed. The behavior of $\mathscr{M}(\varepsilon)$ approximates that of $\mathscr{M}(1)$ and $\mathscr{M}(0)$ for ε tending towards 1 and 0, respectively.

Subsequently, I consider the equilibrium problem of a solid made of a $\mathscr{M}(\varepsilon)$ material and I study the behavior of the solution $[\mathbf{u}(\cdot;\varepsilon), \mathbf{E}(\cdot;\varepsilon), \mathbf{T}(\cdot;\varepsilon)]$ for ε approaching to 1 and 0. A similar problem has been discussed by Wang (1995) who studied the behavior of a linear elastic material with Lamé moduli μ and λ when μ goes to 0 and have provided relationships between the equilibrium states of the material defined by the constitutive relation $\mathbf{T} = 2\mu\mathbf{E} + \lambda(\mathrm{tr}\mathbf{E})\mathbf{I}$, on the one hand, $\mathbf{T} = \lambda(\mathrm{tr}\mathbf{E})\mathbf{I}$, on the other.

In Section 3 of this work, it is shown that the strong monotonicitiy of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ for $\varepsilon > 0$ allows proving that the solution to the equilibrium problem of solids made of a material having constitutive equation $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is unique in terms of displacement, strain and stress, if the solution satisfies appropriate regularity conditions. For a masonry-like material, on the contrary, uniqueness is guaranteed only in terms of stress. Moreover, the solution $[\mathbf{u}(\cdot;\varepsilon), \mathbf{E}(\cdot;\varepsilon), \mathbf{T}(\cdot;\varepsilon)]$ to the equilibrium problem for a $\mathscr{M}(\varepsilon)$ material converges to the solution $[\mathbf{u}(\cdot;1), \mathbf{E}(\cdot;1), \mathbf{T}(\cdot;1)]$ of the same equilibrium problem for a linear elastic material, for ε going to 1. In particular, $\mathbf{T}(\cdot;\varepsilon)$ and $\mathbf{E}(\cdot;\varepsilon)$ converge in the L^2 norm to $\mathbf{T}(\cdot;1)$ and $\mathbf{E}(\cdot;1)$, respectively, and $\mathbf{u}(\cdot;\varepsilon)$ converges to $\mathbf{u}(\cdot;1)$ with respect to the H^1 norm. This result is guaranteed by the strong monotonicity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ for $\varepsilon > 0$, and by the fact that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is continuous with respect to ε for each fixed $\mathbf{E}$.

Subsequently, it is shown that, if $[\mathbf{u}(\cdot;\varepsilon), \mathbf{E}(\cdot;\varepsilon), \mathbf{T}(\cdot;\varepsilon)]$ is the solution to the equilibrium problem of a solid made of material $\mathscr{M}(\varepsilon)$, and $[\mathbf{u}(\cdot;0), \mathbf{E}(\cdot;0), \mathbf{T}(\cdot;0)]$ is a solution to the same equilibrium problem for masonry-like solids, then $\mathbf{T}(\cdot;\varepsilon)$ converges to $\mathbf{T}(\cdot;0)$ in L^2 for ε tending towards 0. This outcome may provide a way to overcome the difficulties encountered during solution of the equilibrium problem for masonry solids via the finite element method. In fact, in order to improve the convergence of the numerical method, it may be convenient to solve an approximate boundary-value problem obtained by substituting a masonry-like material with a $\mathscr{M}(\varepsilon)$ material for ε near to 0.

The results obtained in Section 3 also hold if we forego the assumption of plane strain made here and consider another partition of the strain space and consequently another stress function. In fact, the outcome stems from the strong monotonicity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ for $\varepsilon > 0$.

2. The constitutive equations

In this article, I limit myself to consideration of a plane strain, in other words I consider all strain tensors for which $\mathbf{q}_3$ is the eigenvector corresponding to the zero eigenvalue, where $\mathbf{q}_3$ is a fixed vector. In order to set forth the constitutive equation, let us consider the two-dimensional linear space $\mathscr{V}$ orthogonal to $\mathbf{q}_3$. Let Lin denote the space of all linear applications (second-order tensors) of $\mathscr{V}$ into $\mathscr{V}$ having the inner product $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$, $\mathbf{A}, \mathbf{B} \in \mathrm{Lin}$, with $\mathbf{A}^{\mathrm{T}}$, the transpose of $\mathbf{A}$ and tr, the trace functional. Let us indicate as Sym the subspace of Lin constituted by symmetric tensors.

For the infinitesimal strain $\mathbf{E} \in \mathrm{Sym}$, let e_1 and e_2, with $e_1 \leqslant e_2$, be its eigenvalues corresponding to the eigenvectors $\mathbf{q}_1$ and $\mathbf{q}_2$. We now consider the symmetric tensors

$$\mathbf{O}_1 = \mathbf{q}_1 \otimes \mathbf{q}_1, \qquad \mathbf{O}_2 = \mathbf{q}_2 \otimes \mathbf{q}_2, \qquad \mathbf{O}_3 = \frac{1}{\sqrt{2}}(\mathbf{q}_1 \otimes \mathbf{q}_2 + \mathbf{q}_2 \otimes \mathbf{q}_1), \tag{1}$$

where $\otimes$ denotes the tensor product defined by $(\mathbf{a}\otimes\mathbf{b})\mathbf{v} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a}$, $\forall \mathbf{a}, \mathbf{b}, \mathbf{v} \in \mathscr{V}$. In view of the spectral theorem, we have

$$\mathbf{E} = e_1\mathbf{O}_1 + e_2\mathbf{O}_2. \tag{2}$$

Let μ and λ be the Lame' moduli of the linear elastic material $\mathscr{M}(1)$, with $\lambda > 0$. The case $\lambda = 0$ will be dealt with separately in Appendix A. For $\varepsilon \in [0, 1]$, and $\alpha = \lambda/\mu$, let us define the hypersurfaces of Sym, $\mathscr{I}_1(\varepsilon)$ and $\mathscr{I}_2(\varepsilon)$,

$$\mathscr{I}_1(\varepsilon) = \{\mathbf{E} \in \text{Sym} \mid g_1(\mathbf{E}; \varepsilon) = 0\}, \tag{3}$$

$$\mathscr{I}_2(\varepsilon) = \{\mathbf{E} \in \text{Sym} \mid g_2(\mathbf{E}; \varepsilon) = 0\}, \tag{4}$$

where

$$g_1(\mathbf{E}; \varepsilon) = [(1 - \varepsilon)\alpha + \varepsilon]e_1 + (1 - \varepsilon)(2 + \alpha)e_2, \tag{5}$$

$$g_2(\mathbf{E}; \varepsilon) = e_1 \tag{6}$$

are isotropic functions. Let us define the open cones of Sym

$$\mathscr{R}_1(\varepsilon) = \{\mathbf{E} \in \text{Sym} \mid g_1(\mathbf{E}; \varepsilon) < 0\}, \tag{7}$$
$$\mathscr{R}_2(\varepsilon) = \{\mathbf{E} \in \text{Sym} \mid g_2(\mathbf{E}; \varepsilon) > 0\}, \tag{8}$$
$$\mathscr{R}_3(\varepsilon) = \{\mathbf{E} \in \text{Sym} \mid g_1(\mathbf{E}; \varepsilon) > 0,\ g_2(\mathbf{E}; \varepsilon) < 0\}. \tag{9}$$

Sets $\mathscr{R}_i(\varepsilon)$ are invariant under Orth, the group of all orthogonal tensors. $\mathscr{R}_2(\varepsilon)$ is convex for every $\varepsilon \in [0, 1]$ and $\mathscr{R}_1(\varepsilon)$ is convex only for $\varepsilon \in [0, \frac{2}{3}]$, in fact for these values of ε, $\mathscr{R}_1(\varepsilon) = \mathbb{P}(\varepsilon)^{-1}[\text{Sym}^-]$, where $\mathbb{P}(\varepsilon) = (2 - 3\varepsilon)\mathbb{I} + [(1 - \varepsilon)\alpha + \varepsilon]\mathbf{I} \otimes \mathbf{I}$, and Sym^- is the convex cone of symmetric negative definite tensors. $\mathbb{I}$ is the fourth-order identity tensor over Sym and $\mathbf{A} \otimes \mathbf{B}$, with $\mathbf{A}$ and $\mathbf{B}$ belonging to Sym, is the fourth-order tensor defined by $\mathbf{A} \otimes \mathbf{B}[\mathbf{H}] = (\mathbf{B} \cdot \mathbf{H})\mathbf{A}, \mathbf{H} \in \text{Sym}$. $\mathscr{R}_3(\varepsilon)$ is not convex, in fact, for $\alpha < 2$ and $\varepsilon = \frac{1}{2}$, tensors $\mathbf{E}_1 = \frac{3}{2}\mathbf{O}_1 - 2\mathbf{O}_2$ and $\mathbf{E}_2 = \frac{3}{2}\mathbf{O}_2 - 2\mathbf{O}_1$ belong to $\mathscr{R}_3(\varepsilon)$, but $\mathbf{E}_1 + \mathbf{E}_2 = -\frac{1}{2}\mathbf{I} \notin \mathscr{R}_3(\varepsilon)$.

Denoting by $\overline{\mathscr{R}_i(\varepsilon)}$ the closure of $\mathscr{R}_i(\varepsilon)$ in Sym, we have $\overline{\mathscr{R}_1(\varepsilon)} \cap \overline{\mathscr{R}_3(\varepsilon)} = \mathscr{I}_1(\varepsilon)$ and $\overline{\mathscr{R}_2(\varepsilon)} \cap \overline{\mathscr{R}_3(\varepsilon)} = \mathscr{I}_2(\varepsilon)$. Fig. 1 shows the regions $\overline{\mathscr{R}_1(\varepsilon)}, \overline{\mathscr{R}_2(\varepsilon)}$ and $\overline{\mathscr{R}_3(\varepsilon)}$ represented in the half-plane $\mathscr{E} = \{e_1 \leqslant e_2\}$ of the principal strain plane $e_1 - e_2$. For $\varepsilon = 0$, we have

$$\overline{\mathscr{R}_1(0)} = \{\mathbf{E} \in \text{Sym} \mid \alpha e_1 + (2 + \alpha)e_2 \leqslant 0\}, \tag{10}$$
$$\overline{\mathscr{R}_2(0)} = \{\mathbf{E} \in \text{Sym} \mid e_1 \geqslant 0\}, \tag{11}$$
$$\overline{\mathscr{R}_3(0)} = \{\mathbf{E} \in \text{Sym} \mid \alpha e_1 + (2 + \alpha)e_2 \geqslant 0, e_1 \leqslant 0\} \tag{12}$$

and the three regions, shown in Fig. 2, coincide with those introduced in Lucchesi et al. (1994) for masonry-like materials. For $\varepsilon = 1$, the regions become

$$\overline{\mathscr{R}_1(1)} = \{\mathbf{E} \in \text{Sym} \mid e_1 \leqslant 0\}, \tag{13}$$
$$\overline{\mathscr{R}_2(1)} = \{\mathbf{E} \in \text{Sym} \mid e_1 \geqslant 0\}, \tag{14}$$
$$\overline{\mathscr{R}_3(1)} = \{\mathbf{E} \in \text{Sym} \mid e_1 = 0\}. \tag{15}$$

and are shown in Fig. 3.

For $\varepsilon \in [0, 1]$, let us consider the following function $\widehat{\mathbf{T}}(\mathbf{E}; \varepsilon)$ dependent on $\mathbf{E} \in \text{Sym}$, with values in Sym, delivering the Cauchy stress $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}; \varepsilon)$ corresponding to the infinitesimal strain $\mathbf{E}$,

$$\widehat{\mathbf{T}}(\mathbf{E}; \varepsilon) = 2\mu\mathbf{E} + \lambda\,\text{tr}(\mathbf{E})\mathbf{I}, \quad \mathbf{E} \in \mathscr{R}_1(\varepsilon), \tag{16}$$

$$\widehat{\mathbf{T}}(\mathbf{E}; \varepsilon) = c(\varepsilon)\mathbf{E} + \lambda_2(\varepsilon)\,\text{tr}(\mathbf{E})\mathbf{I}, \quad \mathbf{E} \in \mathscr{R}_2(\varepsilon), \tag{17}$$

$$\widehat{\mathbf{T}}(\mathbf{E}; \varepsilon) = a(\varepsilon)e_1\mathbf{O}_1 + b(\varepsilon)e_2\mathbf{O}_2 + \lambda_3(\varepsilon)\,\text{tr}(\mathbf{E})\mathbf{I}, \quad \mathbf{E} \in \mathscr{R}_3(\varepsilon), \tag{18}$$

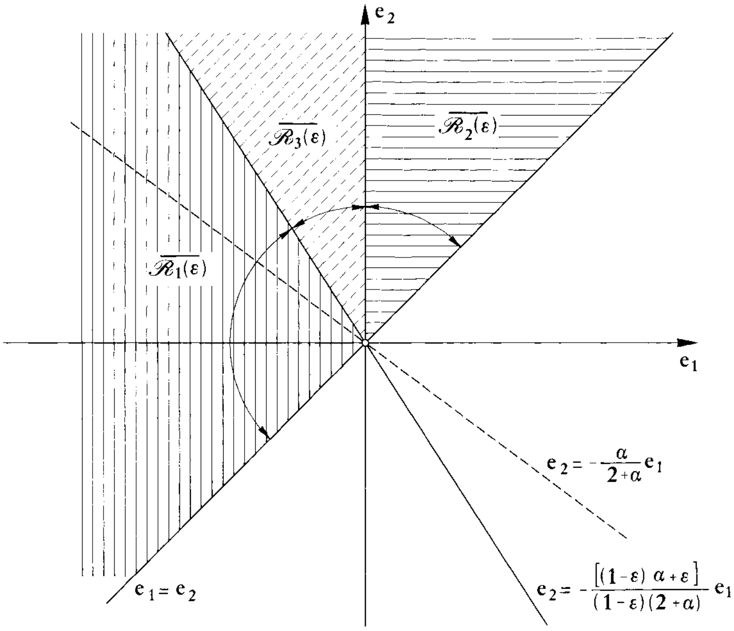

Fig. 1. Regions $\overline{\mathscr{R}_1(\varepsilon)}$, $\overline{\mathscr{R}_2(\varepsilon)}$ and $\overline{\mathscr{R}_3(\varepsilon)}$ for $\varepsilon \in (0, 1)$.

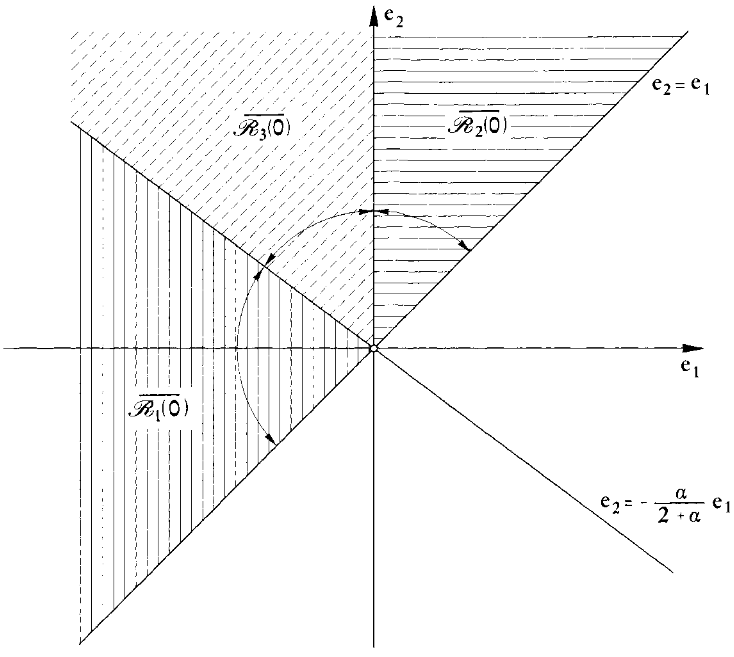

Fig. 2. Regions $\overline{\mathscr{R}_1(0)}$, $\overline{\mathscr{R}_2(0)}$ and $\overline{\mathscr{R}_3(0)}$.

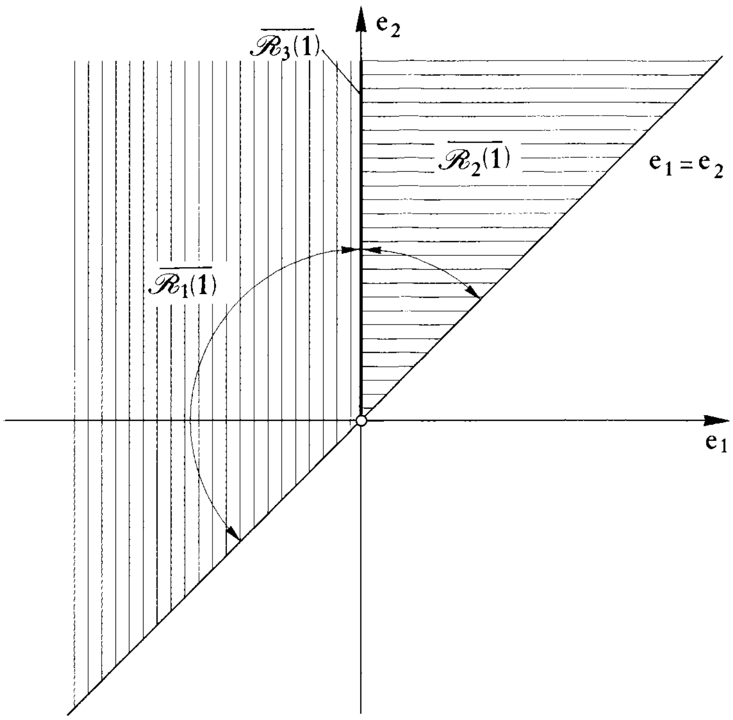

Fig. 3. Regions $\overline{\mathscr{R}_1(1)}$, $\overline{\mathscr{R}_2(1)}$ and $\overline{\mathscr{R}_3(1)}$.

where $\mathbf{I}$ is the second-order identity tensor, and functions $a(\varepsilon)$, $b(\varepsilon)$, $c(\varepsilon)$, $\lambda_2(\varepsilon)$ and $\lambda_3(\varepsilon)$ are to be determined by imposing the continuity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ at the interfaces $\mathscr{I}_1(\varepsilon)$ and $\mathscr{I}_2(\varepsilon)$. Thus, from Eqs. (16)–(18), by accounting for Eqs. (7)–(9) one obtains the following conditions which must be satisfied for every $\varepsilon \in [0,1]$,

$$(1-\varepsilon)(2+\alpha)a(\varepsilon) + (2-3\varepsilon)\lambda_3(\varepsilon) = (1-\varepsilon)(2+\alpha)2\mu + (2-3\varepsilon)\lambda, \tag{19}$$

$$[(1-\varepsilon)\alpha+\varepsilon]b(\varepsilon) - (2-3\varepsilon)\lambda_3(\varepsilon) = [(1-\varepsilon)\alpha+\varepsilon]2\mu - (2-3\varepsilon)\lambda, \tag{20}$$

$$\lambda_2(\varepsilon) = \lambda_3(\varepsilon), \tag{21}$$

$$c(\varepsilon) = b(\varepsilon). \tag{22}$$

Moreover, as for $\varepsilon = 0$, $\widehat{\mathbf{T}}(\mathbf{E};0)$ is required to be the stress for masonries (Lucchesi et al., 1994),

$$\widehat{\mathbf{T}}(\mathbf{E};0) = 2\mu\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{I}, \quad \mathbf{E} \in \mathscr{R}_1(0), \tag{23}$$

$$\widehat{\mathbf{T}}(\mathbf{E};0) = \mathbf{0}, \quad \mathbf{E} \in \mathscr{R}_2(0), \tag{24}$$

$$\widehat{\mathbf{T}}(\mathbf{E};0) = \varphi e_1 \mathbf{O}_1, \quad \mathbf{E} \in \mathscr{R}_3(0), \tag{25}$$

where $\varphi = 4\mu(1+\alpha)/(2+\alpha)$, functions $a(\varepsilon)$, $b(\varepsilon)$, $c(\varepsilon)$, $\lambda_2(\varepsilon)$ and $\lambda_3(\varepsilon)$ have to satisfy the additional conditions

$$a(0) = \varphi, \tag{26}$$
$$b(0) = 0, \tag{27}$$
$$\lambda_3(0) = 0, \tag{28}$$
$$c(0) = 0, \tag{29}$$
$$\lambda_2(0) = 0. \tag{30}$$

Finally, in conformity with the fact that $\widehat{\mathbf{T}}(\mathbf{E};1)$ is the stress for a linear elastic material with Lamé moduli μ and λ, we have:

$$a(1) = 2\mu, \tag{31}$$
$$b(1) = 2\mu, \tag{32}$$
$$\lambda_3(1) = \lambda, \tag{33}$$
$$c(1) = 2\mu, \tag{34}$$
$$\lambda_2(1) = \lambda. \tag{35}$$

In order to obtain $a(\varepsilon)$ and $b(\varepsilon)$ from Eqs. (19) and (20), we need to make some assumptions regarding $\lambda_3(\varepsilon)$. Specifically, we assume that $\lambda_3(\varepsilon)$ is a non-negative quadratic function of ε. Thus, in keeping with Eqs. (28) and (33), we have

$$\lambda_3(\varepsilon) = \lambda\varepsilon(2-\varepsilon). \tag{36}$$

If we assume the linear relationship $\lambda_3(\varepsilon) = \lambda\varepsilon$, a function $a(\varepsilon)$ satisfying both Eqs. (19) and (31) does not exist. Of all possible quadratic relationships, Eq. (36) is the only choice. In fact, if we take $\lambda_3(\varepsilon) = \lambda\varepsilon(\widetilde{\varsigma}\varepsilon + 1 - \widetilde{\varsigma})$, with $\widetilde{\varsigma} \in \mathbb{R}$, and insert $\lambda_3(\varepsilon)$ into Eq. (19), then solve with respect to $a(\varepsilon)$, it is easy to verify that $a(\varepsilon)$ satisfies Eq. (31) if and only if $\widetilde{\varsigma} = -1$, which gives Eq. (36). Therefore, $a(\varepsilon)$ and $b(\varepsilon)$ take the expressions:

$$a(\varepsilon) = 2\mu + \frac{\lambda}{2+\alpha}(1-\varepsilon)(2-3\varepsilon), \tag{37}$$

$$b(\varepsilon) = 2\mu - \lambda\frac{(1-\varepsilon)^2(2-3\varepsilon)}{(1-\varepsilon)\alpha+\varepsilon}. \tag{38}$$

From Eqs. (22) and (21) we get

$$c(\varepsilon) = 2\mu - \lambda\frac{(1-\varepsilon)^2(2-3\varepsilon)}{(1-\varepsilon)\alpha+\varepsilon}, \tag{39}$$

$$\lambda_2(\varepsilon) = \lambda\varepsilon(2-\varepsilon). \tag{40}$$

It is easy to prove that for each $\varepsilon \in [0,1]$,

$$a(\varepsilon) \geqslant 0, \quad b(\varepsilon) \geqslant 0, \quad \lambda_3(\varepsilon) \geqslant 0, \tag{41}$$
$$b(\varepsilon) \leqslant a(\varepsilon) \quad \text{for } \varepsilon \in [0,\tfrac{2}{3}], \qquad a(\varepsilon) \leqslant b(\varepsilon) \quad \text{for } \varepsilon \in [\tfrac{2}{3},1]. \tag{42}$$

A material having constitutive Eqs. (16)–(18) will be denoted by $\mathscr{M}(\varepsilon)$.

The normal stress t_3, corresponding to the vector $\mathbf{q}_3$, is equal to $(\lambda/2(\mu+\lambda))(t_1+t_2)$, where t_1 and t_2 are the eigenvalues of $\mathbf{T}$ corresponding to the eigenvectors $\mathbf{q}_1$ and $\mathbf{q}_2$. In fact, relation $t_3 = (\lambda/2(\mu+\lambda))(t_1+t_2)$ holds in $\mathscr{R}_1(\varepsilon)$, where the material exhibits linear elastic behavior, and is extended by continuity to the other two regions.

For every $\varepsilon \in [0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$, given in Eqs. (16)–(18), is an isotropic, positively homogeneous of degree one, non-linear function. The strain energy density corresponding to $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is

$$\widehat{\psi}(\mathbf{E};\varepsilon) = \mu||\mathbf{E}||^2 + \tfrac{1}{2}\lambda(\mathrm{tr}(\mathbf{E}))^2, \quad \mathbf{E} \in \mathscr{R}_1(\varepsilon), \tag{43}$$

$$\widehat{\psi}(\mathbf{E};\varepsilon) = \tfrac{1}{2}\Big\{b(\varepsilon)||\mathbf{E}||^2 + \lambda_3(\varepsilon)(\mathrm{tr}(\mathbf{E}))^2\Big\}, \quad \mathbf{E} \in \mathscr{R}_2(\varepsilon), \tag{44}$$

$$\widehat{\psi}(\mathbf{E};\varepsilon) = \tfrac{1}{2}\Big\{a(\varepsilon)e_1^2 + b(\varepsilon)e_2^2 + \lambda_3(\varepsilon)(\mathrm{tr}(\mathbf{E}))^2\Big\}, \quad \mathbf{E} \in \mathscr{R}_3(\varepsilon) \tag{45}$$

with $\|\mathbf{E}\| = (\mathbf{E}\cdot\mathbf{E})^{1/2}$; thus, materials $\mathscr{M}(\varepsilon)$ are hyperelastic.

For each $\varepsilon \in (0,1]$, let us consider the fourth-order tensor

$$\mathbb{C}(\varepsilon) = b(\varepsilon)\mathbb{I} + \lambda_3(\varepsilon)\mathbf{I}\otimes\mathbf{I}. \tag{46}$$

The eigenvalues of $\mathbb{C}(\varepsilon)$ are $b(\varepsilon)$ and $b(\varepsilon) + 2\lambda_3(\varepsilon)$, both of which are positive. Therefore $\mathbb{C}(\varepsilon)$ is positive definite and invertible, with inverse

$$\mathbb{C}(\varepsilon)^{-1} = \frac{1}{b(\varepsilon)}\mathbb{I} - \frac{\lambda_3(\varepsilon)}{b(\varepsilon)(b(\varepsilon)+2\lambda_3(\varepsilon))}\mathbf{I}\otimes\mathbf{I}. \tag{47}$$

Some properties of the function $\widehat{\mathbf{T}}$ are collected in the following proposition:

Proposition 1. (*i*) *For* $\varepsilon > 0$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ *is strongly monotone, i.e. there exists a positive scalar* $\kappa(\varepsilon)$ *such that*

$$(\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon))\cdot(\mathbf{E}_1 - \mathbf{E}_2) \geqslant \kappa(\varepsilon)||\mathbf{E}_1 - \mathbf{E}_2||^2 \quad \forall\ \mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}. \tag{48}$$

(*ii*) *For* $\varepsilon \in [0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ *is monotone, in particular, it holds that*

$$(\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon))\cdot(\mathbf{E}_1 - \mathbf{E}_2) \geqslant \frac{1}{2(\mu+\lambda)}||\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon)||^2 \quad \forall\ \mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}. \tag{49}$$

(*iii*) *For* $\varepsilon \in [0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ *is Lipschitz continuous*

$$||\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon)|| \leqslant 2(\mu+\lambda)||\mathbf{E}_1 - \mathbf{E}_2|| \quad \forall\ \mathbf{E}_1, \mathbf{E}_2 \in \mathrm{Sym}. \tag{50}$$

Proof. (i) Let us start by proving that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is strongly monotone in the three regions separately. If $\mathbf{E}_1, \mathbf{E}_2 \in \overline{\mathscr{R}_1(\varepsilon)}$, then, from Eq. (16) it follows that

$$(\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon))\cdot(\mathbf{E}_1 - \mathbf{E}_2) \geqslant 2\mu||\mathbf{E}_1 - \mathbf{E}_2||^2; \tag{51}$$

and if $\mathbf{E}_1, \mathbf{E}_2 \in \overline{\mathscr{R}_2(\varepsilon)}$, from Eq. (17) we get

$$(\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon))\cdot(\mathbf{E}_1 - \mathbf{E}_2) \geqslant b(\varepsilon)||\mathbf{E}_1 - \mathbf{E}_2||^2. \tag{52}$$

Now, we have to prove that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is strongly monotone in the non-convex cone $\overline{\mathscr{R}_3(\varepsilon)}$. Let us set

$$\kappa(\varepsilon) = \begin{cases} b(\varepsilon) & \text{for } \varepsilon \in (0, \frac{2}{3}], \\ \eta_2(\varepsilon) & \text{for } \varepsilon \in [\frac{2}{3}, 1] \end{cases} \tag{53}$$

with

$$\eta_2(\varepsilon) = \tfrac{1}{2}\Big\{a(\varepsilon) + b(\varepsilon) + 2\lambda_3(\varepsilon) - \sqrt{(a(\varepsilon)-b(\varepsilon))^2 + 4\lambda_3(\varepsilon)^2}\Big\}; \tag{54}$$

because of Eqs. (41) and (42), $\eta_2(\varepsilon)$ is positive for $\varepsilon > 0$ and zero for $\varepsilon = 0$, moreover we have

$$a(\varepsilon) \leqslant \eta_2(\varepsilon) \leqslant b(\varepsilon) \quad \text{for } \varepsilon \in [\tfrac{2}{3}, 1]. \tag{55}$$

Given $\mathbf{E}_1, \mathbf{E}_2 \in \mathscr{R}_3(\varepsilon)$, let $\mathbf{E}_1 = e_1\mathbf{O}_1 + e_2\mathbf{O}_2$ and $\mathbf{E}_2 = u_1\mathbf{P}_1 + u_2\mathbf{P}_2$ be their spectral representations, where $\mathbf{O}_i = \mathbf{q}_i \otimes \mathbf{q}_i$, $\mathbf{P}_i = \mathbf{p}_i \otimes \mathbf{p}_i$, $i = 1, 2$ and $\mathbf{q}_i$ and $\mathbf{p}_i$ are the eigenvectors of $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. Simple calculations show that strong monotonicity (48) is equivalent to the inequality

$$\begin{aligned}&(a(\varepsilon) - \kappa(\varepsilon))\|e_1\mathbf{O}_1 - u_1\mathbf{P}_1\|^2 + (b(\varepsilon) - \kappa(\varepsilon))\|e_2\mathbf{O}_2 - u_2\mathbf{P}_2\|^2 - (a(\varepsilon) + b(\varepsilon) - 2\kappa(\varepsilon))\\&\quad \times (e_1\mathbf{O}_1 \cdot u_2\mathbf{P}_2 + e_2\mathbf{O}_2 \cdot u_1\mathbf{P}_1) + \lambda_3(\varepsilon)(\mathrm{tr}(\mathbf{E}_1) - \mathrm{tr}(\mathbf{E}_2))^2 \geqslant 0.\end{aligned} \tag{56}$$

As in $\overline{\mathscr{R}_3(\varepsilon)}$ we have $e_1, u_1 \leqslant 0$ and $e_2, u_2 \geqslant 0$, in view of Eqs. (53) and (55), Eq. (56) is satisfied if

$$(a(\varepsilon) - \kappa(\varepsilon))\|e_1\mathbf{O}_1 - u_1\mathbf{P}_1\|^2 + (b(\varepsilon) - \kappa(\varepsilon))\|e_2\mathbf{O}_2 - u_2\mathbf{P}_2\|^2 + \lambda_3(\varepsilon)(\mathrm{tr}(\mathbf{E}_1) - \mathrm{tr}(\mathbf{E}_2))^2 \geqslant 0. \tag{57}$$

If $\varepsilon \in (0, \frac{2}{3}]$, the left-hand side of Eq. (57) is the sum of non-negative quantities; if $\varepsilon \in [\frac{2}{3}, 1)$, the proof of inequality (57) requires some calculations. Let $\mathbf{Q} \in \mathrm{Orth}$ be the orthogonal tensor such that $\mathbf{p}_i = \mathbf{Q}\mathbf{q}_i$. Then, for $\cos^2\delta = (\mathbf{q}_1 \cdot \mathbf{Q}\mathbf{q}_1)^2 = (\mathbf{q}_2 \cdot \mathbf{Q}\mathbf{q}_2)^2$, and $\sin^2\delta = (\mathbf{q}_1 \cdot \mathbf{Q}\mathbf{q}_2)^2 = (\mathbf{q}_2 \cdot \mathbf{Q}\mathbf{q}_1)^2$, elementary calculations show that Eq. (57) is equivalent to the inequality

$$\begin{aligned}&\{(a(\varepsilon) - \eta_2(\varepsilon))(e_1 - u_1)^2 + (b(\varepsilon) - \eta_2(\varepsilon))(e_2 - u_2)^2 + \lambda_3(\varepsilon)((e_1 - u_1)^2 + (e_2 - u_2)^2\\&\quad + 2(e_1 - u_1)(e_2 - u_2))\}\cos^2\delta + \{(a(\varepsilon) - \eta_2(\varepsilon))(e_1^2 + u_1^2) + (b(\varepsilon) - \eta_2(\varepsilon))(e_2^2 + u_2^2)\\&\quad + \lambda_3(\varepsilon)((e_1 - u_1)^2 + (e_2 - u_2)^2 + 2(e_1 - u_1)(e_2 - u_2))\}\sin^2\delta.\end{aligned} \tag{58}$$

As far as the coefficient of $\cos^2\delta$ is concerned, if $e_1 - u_1 = 0$, then it is non-negative. Otherwise, for $e_1 - u_1 \neq 0$, it has the same sign of the parabola

$$p_0(z) = (b(\varepsilon) + \lambda_3(\varepsilon) - \eta_2(\varepsilon))z^2 + 2\lambda_3(\varepsilon)z + a(\varepsilon) + \lambda_3(\varepsilon) - \eta_2(\varepsilon), \tag{59}$$

where $z = (e_2 - u_2)/(e_1 - u_1)$. As the vertex of p_0 has coordinates $(-(\lambda_3(\varepsilon))/(b(\varepsilon) + \lambda_3(\varepsilon) - \eta_2(\varepsilon)), 0)$ and the quantity $b(\varepsilon) + \lambda_3(\varepsilon) - \eta_2(\varepsilon)$ is greater than 0, then $p_0(z) \geqslant 0$ for each $z \in \mathbb{R}$. The coefficient of $\sin^2\delta$ is the sum of the coefficient of $\cos^2\delta$ plus the quantity

$$2(a(\varepsilon) - \eta_2(\varepsilon))e_1u_1 + 2(b(\varepsilon) - \eta_2(\varepsilon))e_2u_2, \tag{60}$$

which is non-negative. In fact, in $\overline{\mathscr{R}_3(\varepsilon)}$, we have $e_1 = he_2$ and $u_1 = ku_2$, with $h, k \in [-((2 + \alpha)(1 - \varepsilon))/((1 - \varepsilon)\alpha + \varepsilon), 0]$, and the expression in Eq. (60) has the same sign of the function

$$\widetilde{p}(h, k) = (a(\varepsilon) - \eta_2(\varepsilon))hk + b(\varepsilon) - \eta_2(\varepsilon), \tag{61}$$

that in the square $[-((2 + \alpha)(1 - \varepsilon))/((1 - \varepsilon)\alpha + \varepsilon), 0] \times [-((2 + \alpha)(1 - \varepsilon))/((1 - \varepsilon)\alpha + \varepsilon), 0]$ is non-negative, as can be checked through simple calculations.

As for every $\varepsilon \in (0, 1]$, the inequalities $2\mu \geqslant \kappa(\varepsilon)$ and $b(\varepsilon) \geqslant \kappa(\varepsilon)$ hold, we deduce that Eq. (48) is satisfied in each region $\overline{\mathscr{R}_i(\varepsilon)}$.

Now, let us suppose that $\mathbf{E}_1$ belongs to $\mathscr{R}_1(\varepsilon)$, $(g_1(\mathbf{E}_1; \varepsilon) \neq 0)$ and $\mathbf{E}_2$ belongs to $\mathscr{R}_3(\varepsilon)$, in particular, $g_1(\mathbf{E}_1; \varepsilon) < 0$ and $g_1(\mathbf{E}_2; \varepsilon) > 0$. Let us consider the continuous function $\widetilde{g}_1(t) = g_1((1 - t)\mathbf{E}_1 + t\mathbf{E}_2; \varepsilon)$, $t \in [0, 1]$. We have $\widetilde{g}_1(0) < 0$ and $\widetilde{g}_1(1) > 0$, and therefore there exists $\bar{t} \in (0, 1)$, such that $\widetilde{g}_1(\bar{t}) = 0$. Let us put $\mathbf{K} = (1 - \bar{t})\mathbf{E}_1 + \bar{t}\mathbf{E}_2$. $\mathbf{K}$ belongs to $\mathscr{I}_1(\varepsilon) = \overline{\mathscr{R}_1(\varepsilon)} \cap \overline{\mathscr{R}_3(\varepsilon)}$, and therefore as Eq. (48) holds in every region, we have:

$$(\widehat{\mathbf{T}}(\mathbf{E}_1; \varepsilon) - \widehat{\mathbf{T}}(\mathbf{K}; \varepsilon)) \cdot (\mathbf{E}_1 - \mathbf{K}) \geqslant \kappa(\varepsilon)\|\mathbf{E}_1 - \mathbf{K}\|^2, \tag{62}$$

$$(\widehat{\mathbf{T}}(\mathbf{K}; \varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_2; \varepsilon)) \cdot (\mathbf{K} - \mathbf{E}_2) \geqslant \kappa(\varepsilon)\|\mathbf{K} - \mathbf{E}_2\|^2. \tag{63}$$

By accounting for $\mathbf{E}_1 - \mathbf{K} = \bar{t}(\mathbf{E}_1 - \mathbf{E}_2)$ and $\mathbf{K} - \mathbf{E}_2 = (1-\bar{t})(\mathbf{E}_1 - \mathbf{E}_2)$, Eqs. (62) and (63) can be rewritten as follows:

$$\widehat{\mathbf{T}}(\mathbf{E}_1;\varepsilon)\cdot(\mathbf{E}_1-\mathbf{E}_2) \geqslant \widehat{\mathbf{T}}(\mathbf{K};\varepsilon)\cdot(\mathbf{E}_1-\mathbf{E}_2) + \kappa(\varepsilon)\bar{t}||\mathbf{E}_1-\mathbf{E}_2||^2, \tag{64}$$

$$\widehat{\mathbf{T}}(\mathbf{K};\varepsilon)\cdot(\mathbf{E}_1-\mathbf{E}_2) \geqslant \widehat{\mathbf{T}}(\mathbf{E}_2;\varepsilon)\cdot(\mathbf{E}_1-\mathbf{E}_2) + \kappa(\varepsilon)(1-\bar{t})||\mathbf{E}_1-\mathbf{E}_2||^2. \tag{65}$$

Substituting the latter in the former yields Eq. (48).

We proceed in an analogous way if $\mathbf{E}_1$ belongs to $\mathscr{R}_1(\varepsilon)$ and $\mathbf{E}_2$ lies in $\overline{\mathscr{R}_3(\varepsilon)}$, with $g_1(\mathbf{E}_2;\varepsilon)\neq 0$; if $\mathbf{E}_1$ belongs to $\overline{\mathscr{R}_3(\varepsilon)}$, with $g_2(\mathbf{E}_1;\varepsilon)\neq 0$, and $\mathbf{E}_2$ lies in $\mathscr{R}_2(\varepsilon)$; if $\mathbf{E}_1$ belongs to $\mathscr{R}_3(\varepsilon)$ and $\mathbf{E}_2$ lies in $\overline{\mathscr{R}_2(\varepsilon)}$, with $g_2(\mathbf{E}_2;\varepsilon)\neq 0$.

We must still consider the final case, in which $\mathbf{E}_1$ belongs to $\overline{\mathscr{R}_1(\varepsilon)}$ with $g_1(\mathbf{E}_1;\varepsilon)\neq 0$ and $\mathbf{E}_2$ lies in $\mathscr{R}_2(\varepsilon)$. As $g_1(\mathbf{E}_1;\varepsilon)<0$ and $g_1(\mathbf{E}_2;\varepsilon)>0$, there exists $\hat{t}\in(0,1)$ such that the tensor $\mathbf{J}=(1-\hat{t})\mathbf{E}_1+\hat{t}\,\mathbf{E}_2$ belongs to the interface $\mathscr{I}_1(\varepsilon)$. Inequality (48) now follows from the fact that monotonicity holds separately for $\mathbf{E}_1$, $\mathbf{J}\in\overline{\mathscr{R}_3}$, for $\mathbf{J}\in\overline{\mathscr{R}_3}$, $\mathbf{E}_2\in\mathscr{R}_2(\varepsilon)$ and from the equalities $\mathbf{E}_1-\mathbf{J}=\hat{t}(\mathbf{E}_1-\mathbf{E}_2)$ and $\mathbf{J}-\mathbf{E}_2=(1-\hat{t})(\mathbf{E}_1-\mathbf{E}_2)$. This allows us to conclude the proof of the strong monotonicity of $\widehat{\mathbf{T}}$ in Sym and therefore of the strict convexity of the strain energy density $\widehat{\psi}$ defined in Eqs. (43)–(45).

(ii) Eq. (49) has been already proved in Del Piero (1989) for $\varepsilon=0$. We must now consider the case $\varepsilon>0$. From Eq. (48), it follows that $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is injective and its inverse $\widehat{\mathbf{T}}^{-1}(\cdot;\varepsilon)$ exists. For t_1 and t_2, the eigenvalues of $\mathbf{T}$ with $t_1\leqslant t_2$, we have

$$\widehat{\mathbf{T}}^{-1}(\mathbf{T};\varepsilon)=\mathbb{C}(1)^{-1}[\mathbf{T}]\quad \text{for } \mathbf{T}\in\mathscr{S}_1(\varepsilon), \tag{66}$$

$$\widehat{\mathbf{T}}^{-1}(\mathbf{T};\varepsilon)=\mathbb{C}(\varepsilon)^{-1}[\mathbf{T}]\quad \text{for } \mathbf{T}\in\mathscr{S}_2(\varepsilon), \tag{67}$$

$$\widehat{\mathbf{T}}^{-1}(\mathbf{T};\varepsilon)=A(\varepsilon)t_1\mathbf{O}_1+B(\varepsilon)t_2\mathbf{O}_2+C(\varepsilon)\,\mathrm{tr}(\mathbf{T})\mathbf{I},\quad \mathbf{T}\in\mathscr{S}_3(\varepsilon), \tag{68}$$

where

$$A(\varepsilon)=\frac{b(\varepsilon)+2\lambda_3(\varepsilon)}{a(\varepsilon)b(\varepsilon)+\lambda_3(\varepsilon)(a(\varepsilon)+b(\varepsilon))}, \tag{69}$$

$$B(\varepsilon)=\frac{a(\varepsilon)+2\lambda_3(\varepsilon)}{a(\varepsilon)b(\varepsilon)+\lambda_3(\varepsilon)(a(\varepsilon)+b(\varepsilon))}, \tag{70}$$

$$C(\varepsilon)=-\frac{\lambda_3(\varepsilon)}{a(\varepsilon)b(\varepsilon)+\lambda_3(\varepsilon)(a(\varepsilon)+b(\varepsilon))}, \tag{71}$$

$$\mathscr{S}_1(\varepsilon)=\{\mathbf{T}\in\mathrm{Sym}\mid h_1(\mathbf{T};\varepsilon)<0\}, \tag{72}$$

$$\mathscr{S}_2(\varepsilon)=\{\mathbf{T}\in\mathrm{Sym}\mid h_2(\mathbf{T};\varepsilon)>0\}, \tag{73}$$

$$\mathscr{S}_3(\varepsilon)=\{\mathbf{T}\in\mathrm{Sym}\mid h_1(\mathbf{T};\varepsilon)>0,\ h_2(\mathbf{T};\varepsilon)<0\} \tag{74}$$

with

$$h_1(\mathbf{T};\varepsilon)=(2+\alpha)\varepsilon t_1+(4(1-\varepsilon)+\alpha(4-5\varepsilon))t_2, \tag{75}$$

$$h_2(\mathbf{T};\varepsilon)=(b(\varepsilon)+\lambda_3(\varepsilon))t_1-\lambda_3(\varepsilon)t_2. \tag{76}$$

Regions $\overline{\mathscr{S}_1(\varepsilon)}$, $\overline{\mathscr{S}_2(\varepsilon)}$ and $\overline{\mathscr{S}_3(\varepsilon)}$ are shown in Figs. 4–6 for $\varepsilon\in(0,(4(1+\alpha))/(4+5\alpha))$, $\varepsilon=(4(1+\alpha))/(4+5\alpha)$ and $\varepsilon\in(4(1+\alpha)/(4+5\alpha),1]$, respectively.

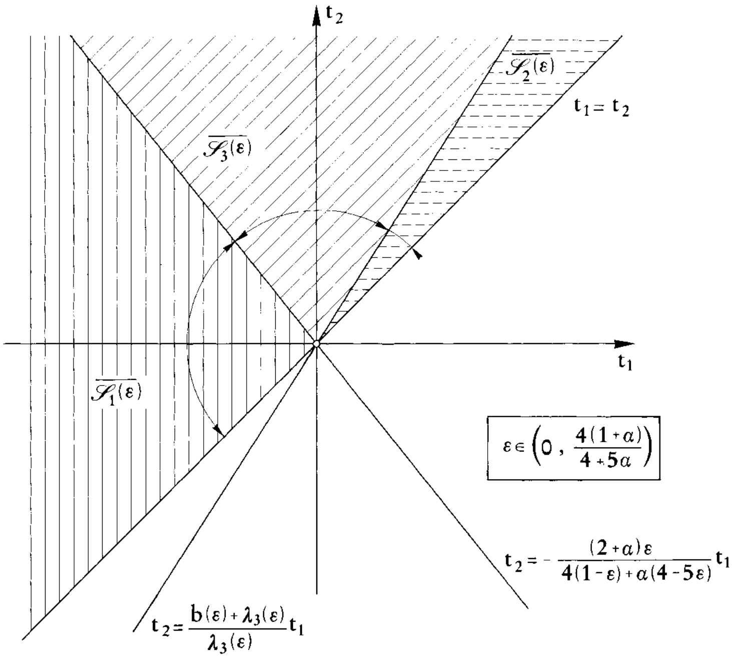

Fig. 4. Regions $\overline{\mathscr{S}_1(\varepsilon)}$, $\overline{\mathscr{S}_2(\varepsilon)}$ and $\overline{\mathscr{S}_3(\varepsilon)}$ for $\varepsilon \in (0, 4(1+\alpha)/(4+5\alpha))$.

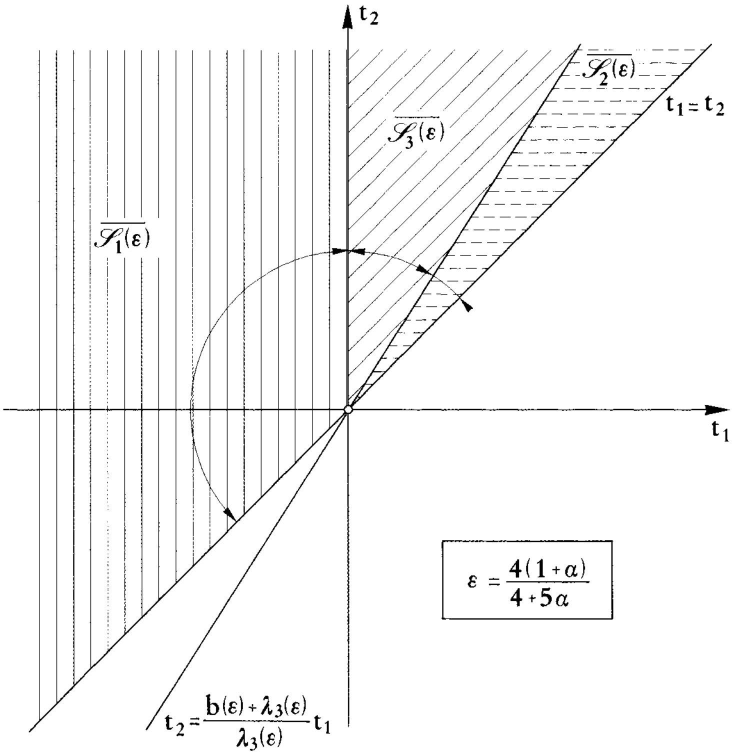

Fig. 5. Regions $\overline{\mathscr{S}_1(\varepsilon)}$, $\overline{\mathscr{S}_2(\varepsilon)}$ and $\overline{\mathscr{S}_3(\varepsilon)}$ for $\varepsilon = 4(1+\alpha)/(4+5\alpha)$.

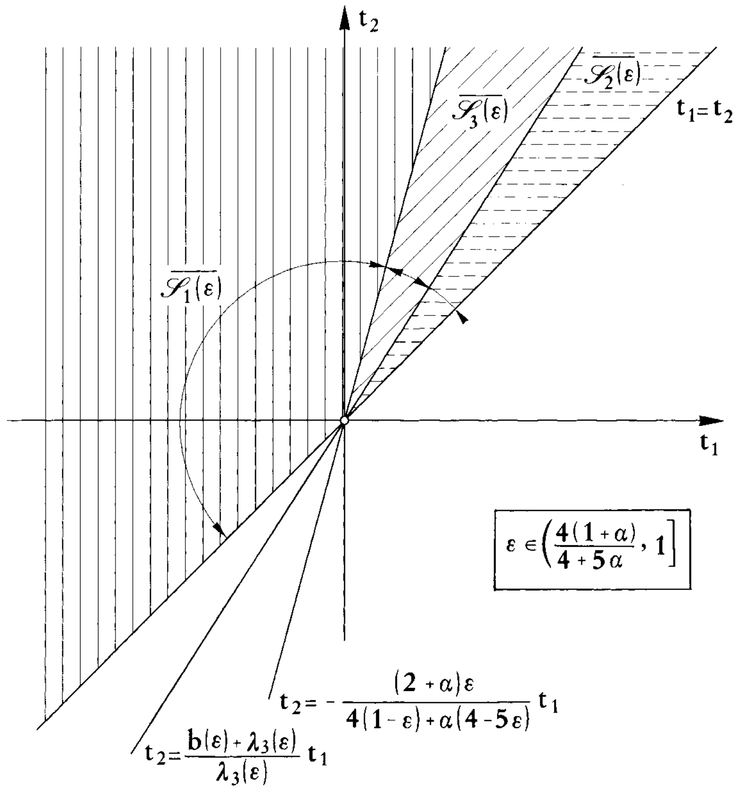

Fig. 6. Regions $\overline{\mathscr{S}_1(\varepsilon)}$, $\overline{\mathscr{S}_2(\varepsilon)}$ and $\overline{\mathscr{S}_3(\varepsilon)}$ for $\varepsilon \in ((4(1+\alpha))/(4+5\alpha), 1]$.

As in the proof of (i), we start by demonstrating that $\widehat{\mathbf{T}}^{-1}(\mathbf{T};\varepsilon)$ is strongly monotone separately in the three regions. If $\mathbf{T}_1, \mathbf{T}_2 \in \overline{\mathscr{S}_1(\varepsilon)}$, then

$$(\widehat{\mathbf{T}}^{-1}(\mathbf{T}_1;\varepsilon) - \widehat{\mathbf{T}}^{-1}(\mathbf{T}_2;\varepsilon)) \cdot (\mathbf{T}_1 - \mathbf{T}_2) \geqslant \frac{1}{2(\mu+\lambda)} ||\mathbf{T}_1 - \mathbf{T}_2||^2 \tag{77}$$

if $\mathbf{T}_1, \mathbf{T}_2 \in \overline{\mathscr{S}_2(\varepsilon)}$, we have

$$(\widehat{\mathbf{T}}^{-1}(\mathbf{T}_1;\varepsilon) - \widehat{\mathbf{T}}^{-1}(\mathbf{T}_2;\varepsilon)) \cdot (\mathbf{T}_1 - \mathbf{T}_2) \geqslant \frac{1}{b(\varepsilon)+2\lambda_3(\varepsilon)} ||\mathbf{T}_1 - \mathbf{T}_2||^2 \geqslant \frac{1}{2(\mu+\lambda)} ||\mathbf{T}_1 - \mathbf{T}_2||^2, \tag{78}$$

where the last inequality comes from Eqs. (36) and (38). Now, let us prove that for $\mathbf{T}_1, \mathbf{T}_2 \in \overline{\mathscr{S}_3(\varepsilon)}$, we have

$$(\widehat{\mathbf{T}}^{-1}(\mathbf{T}_1;\varepsilon) - \widehat{\mathbf{T}}^{-1}(\mathbf{T}_2;\varepsilon)) \cdot (\mathbf{T}_1 - \mathbf{T}_2) \geqslant \frac{1}{2(\mu+\lambda)} ||\mathbf{T}_1 - \mathbf{T}_2||^2. \tag{79}$$

Let $\mathbf{T}_1 = t_1\mathbf{O}_1 + t_2\mathbf{O}_2$ and $\mathbf{T}_2 = s_1\mathbf{P}_1 + s_2\mathbf{P}_2$ be the spectral representations of $\mathbf{T}_1$ and $\mathbf{T}_2$, where $\mathbf{O}_i = \mathbf{q}_i \otimes \mathbf{q}_i$, $\mathbf{P}_i = \mathbf{p}_i \otimes \mathbf{p}_i$, $i = 1, 2$ and $\mathbf{q}_i$ and $\mathbf{p}_i$ are the eigenvectors of $\mathbf{T}_1$ and $\mathbf{T}_2$, respectively. Let $\mathbf{Q} \in \mathrm{Orth}$ be the orthogonal tensor such that $\mathbf{p}_i = \mathbf{Q}\mathbf{q}_i$. Then, for $\cos^2\delta = (\mathbf{q}_1 \cdot \mathbf{Q}\mathbf{q}_1)^2 = (\mathbf{q}_2 \cdot \mathbf{Q}\mathbf{q}_2)^2$, and $\sin^2\delta = (\mathbf{q}_1 \cdot \mathbf{Q}\mathbf{q}_2)^2 = (\mathbf{q}_2 \cdot \mathbf{Q}\mathbf{q}_1)^2$, elementary calculations show that Eq. (79) is equivalent to the inequality

$$\left\{\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)(t_1-s_1)^2+\left(B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)(t_2-s_2)^2+2C(\varepsilon)(t_1-s_1)\right.$$
$$\left.\times(t_2-s_2)\right\}\cos^2\delta+\left\{\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)(t_1^2+s_1^2)+\left(B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)(t_2^2+s_2^2)\right.$$
$$\left.-\left(A(\varepsilon)+B(\varepsilon)+2C(\varepsilon)-\frac{1}{(\mu+\lambda)}\right)(t_1s_2+t_2s_1)+2C(\varepsilon)(t_1-s_2)(t_2-s_1)\right\}\sin^2\delta\geqslant 0. \tag{80}$$

The coefficient of $\cos^2\delta$ is non-negative; this is guaranteed by the fact that for $t_2-s_2=0$, we have

$$A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\geqslant 0\quad\forall\varepsilon\in(0,1] \tag{81}$$

and for $t_2-s_2\neq 0$, the parabola

$$p_1(z)=\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)z^2+2C(\varepsilon)z+B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)} \tag{82}$$

with $z=(t_1-s_1)/(t_2-s_2)$, is positive for each z. By setting $t_1=m(\varepsilon)t_2$, $s_1=n(\varepsilon)s_2$, with $m(\varepsilon)$, $n(\varepsilon)\in[(4(1-\varepsilon)+\alpha(4-5\varepsilon))/(2+\alpha)\varepsilon, \lambda_3(\varepsilon)/(b(\varepsilon)+\lambda_3(\varepsilon))]$, $\varepsilon>0$, the coefficient of $\sin^2\delta$ can be rewritten as

$$\left\{\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)m^2(\varepsilon)+2C(\varepsilon)m(\varepsilon)+B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right\}t_2^2$$
$$+\left\{\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)n^2(\varepsilon)+2C(\varepsilon)n(\varepsilon)+B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right\}s_2^2$$
$$-\left\{\left(A(\varepsilon)+B(\varepsilon)+2C(\varepsilon)-\frac{1}{\mu+\lambda}\right)(m(\varepsilon)+n(\varepsilon))+2C(\varepsilon)(1+m(\varepsilon)n(\varepsilon))\right\}t_2s_2. \tag{83}$$

For $w=s_2/t_2$, with $t_2>0$, the non-negativeness of Eq. (83) is equivalent to the non-negativeness of the parabola

$$p_2(w)=\left\{\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)n^2(\varepsilon)+2C(\varepsilon)n(\varepsilon)+B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right\}w^2$$
$$-\left\{\left(A(\varepsilon)+B(\varepsilon)+2C(\varepsilon)-\frac{1}{\mu+\lambda}\right)(m(\varepsilon)+n(\varepsilon))+2C(\varepsilon)(1+m(\varepsilon)n(\varepsilon))\right\}w$$
$$+\left(A(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}\right)m^2(\varepsilon)+2C(\varepsilon)m(\varepsilon)+B(\varepsilon)+C(\varepsilon)-\frac{1}{2(\mu+\lambda)}. \tag{84}$$

Through calculations omitted here for the sake of brevity, it can be proved that for each $\varepsilon\in(0,1]$, $p_2(w)$ is positive for $w>0$, and the proof of Eq. (79) is thus concluded. Finally, from Eqs. (77)–(79) with arguments similar to those used in proving (i), we get Eq. (49).

(iii) Relation (50) follows directly from Eq. (49) by using the Schwarz inequality. □

From the proof of Proposition 1, a more general result already proved in Curnier et al. (1995) follows, that a continuous piecewise (strongly) monotone function is (strongly) monotone globally, as well.

The next proposition deals with the differentiability of $\widehat{\mathbf{T}}$ with respect to $\mathbf{E}$.

Proposition 2. *For* $\varepsilon\in[0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ *is differentiable in each region* $\mathscr{R}_i(\varepsilon)$,

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)=\mathbb{C}(1)\quad\textit{for }\mathbf{E}\in\mathscr{R}_1(\varepsilon), \tag{85}$$

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = \mathbb{C}(\varepsilon) \quad \textit{for } \mathbf{E} \in \mathscr{R}_2(\varepsilon), \tag{86}$$

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = (a(\varepsilon)+\lambda_3(\varepsilon))\mathbf{O}_1\otimes\mathbf{O}_1 + (b(\varepsilon)+\lambda_3(\varepsilon))\mathbf{O}_2\otimes\mathbf{O}_2 + \lambda_3(\varepsilon)(\mathbf{O}_1\otimes\mathbf{O}_2+\mathbf{O}_2\otimes\mathbf{O}_1),$$

$$\frac{b(\varepsilon)e_2 - a(\varepsilon)e_1}{e_2 - e_1}\mathbf{O}_3\otimes\mathbf{O}_3 \quad \textit{for } \mathbf{E} \in \mathscr{R}_3(\varepsilon). \tag{87}$$

Moreover, $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ *is positive definite for* $\varepsilon > 0$ *and positive semi-definite for* $\varepsilon = 0$.

Proof. The derivative $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ with respect to $\mathbf{E}$ can be explicitly calculated from Eqs. (16)–(18) by using the results provided in Lucchesi et al. (1996). Here, we prove that $D_E\widehat{\mathbf{T}}(\mathbf{E};0)$ is positive semi-definite in each of the three regions $\mathscr{R}_i(\varepsilon)$, while $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$, on the contrary is positive definite in $\mathscr{R}_i(\varepsilon)$ for $\varepsilon > 0$. In fact, if $\mathbf{E} \in \mathscr{R}_1(\varepsilon)$, then for every $\mathbf{H}$ in Sym we have

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)[\mathbf{H}]\cdot\mathbf{H} \geqslant 2\mu||\mathbf{H}||^2, \tag{88}$$

and if $\mathbf{E} \in \mathscr{R}_2(\varepsilon)$, for every $\mathbf{H}$ in Sym we have

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)[\mathbf{H}]\cdot\mathbf{H} \geqslant b(\varepsilon)||\mathbf{H}||^2. \tag{89}$$

Now we have to consider the case of $\mathbf{E} \in \mathscr{R}_3(\varepsilon)$. From Eq. (87), it follows that

$$\eta_1(\mathbf{E};\varepsilon) = \frac{b(\varepsilon)e_2 - a(\varepsilon)e_1}{e_2 - e_1} \tag{90}$$

is the eigenvalue corresponding to $\mathbf{O}_3$. In $\mathscr{R}_3(\varepsilon)$, we have $-(1-\varepsilon)(2+\alpha)/((1-\varepsilon)\alpha+\varepsilon)\, e_2 < e_1 < 0$ and can thus write $e_1 = ke_2$, with $k \in (-(1-\varepsilon)(2+\alpha)/((1-\varepsilon)\alpha+\varepsilon), 0)$. Then, Eq. (90) becomes

$$\eta_1 = \frac{b(\varepsilon) - a(\varepsilon)k}{1-k}. \tag{91}$$

Now, let us consider the function $\eta_1(k;\varepsilon)$ when k lies in $(-(1-\varepsilon)(2+\alpha)/((1-\varepsilon)\alpha+\varepsilon), 0)$: $\eta_1(0;\varepsilon) = b(\varepsilon)$ and $\eta_1(-(1-\varepsilon)(2+\alpha)/((1-\varepsilon)\alpha+\varepsilon);\varepsilon) = 2\mu$. Further, the derivative of $\eta_1(k;\varepsilon)$ with respect to k is negative if $\varepsilon \in [0,\frac{2}{3})$, zero for $\varepsilon = \frac{2}{3}$, and positive if $\varepsilon \in (\frac{2}{3}, 1]$. Therefore, putting $\bar{k} = -(1-\varepsilon)(2+\alpha)/((1-\varepsilon)\alpha+\varepsilon)$, we have

$$\widehat{\eta}_1(\varepsilon) = \inf_{k\in(\bar{k},0)} \eta_1(k;\varepsilon) = \begin{cases} b(\varepsilon) & \text{if } \varepsilon \in [0,\frac{2}{3}), \\ 2\mu & \text{if } \varepsilon \in [\frac{2}{3}, 1], \end{cases} \tag{92}$$

and we see that Eq. (90) is positive for $\varepsilon \geqslant 0$. The other two eigenvalues corresponding to the eigenvectors belonging to Span$\{\mathbf{O}_1, \mathbf{O}_2\}$, are the roots of the second-order equation

$$\eta^2 - (a(\varepsilon) + b(\varepsilon) + 2\lambda_3(\varepsilon))\eta + a(\varepsilon)b(\varepsilon) + \lambda_3(\varepsilon)(a(\varepsilon) + b(\varepsilon)) = 0, \tag{93}$$

one root is $\eta_2(\varepsilon)$ given in Eq. (54) and the other one is

$$\eta_3(\varepsilon) = \tfrac{1}{2}\left\{a(\varepsilon) + b(\varepsilon) + \lambda_3(\varepsilon) + \sqrt{(a(\varepsilon) - b(\varepsilon))^2 + 4\lambda_3(\varepsilon)^2}\right\}. \tag{94}$$

Because of Eqs. (41) and (42), it holds that $\eta_2(\varepsilon) < \eta_3(\varepsilon)$ and $\eta_3(\varepsilon)$ is positive for each $\varepsilon \in [0, 1]$. Due to the fact that $b(\varepsilon) \leqslant \eta_2(\varepsilon)$, if $\varepsilon \in [0,\frac{2}{3}]$, and $2\mu \geqslant \eta_2(\varepsilon)$ for $\varepsilon \in [\frac{2}{3}, 1]$, recalling the definition of $\kappa(\varepsilon)$ given in Eq. (53), we conclude that for $\mathbf{E} \in \mathscr{R}_3(\varepsilon)$, we have

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)[\mathbf{H}]\cdot\mathbf{H} \geqslant \kappa(\varepsilon)||\mathbf{H}||^2 \quad \text{for every } \mathbf{H} \in \text{Sym}. \tag{95}$$

By accounting for Eqs. (53), (88) and (89), we show that condition (95) is satisfied for every $\mathbf{E}$ belonging to $\mathscr{R}_i(\varepsilon)$. Function $\kappa(\varepsilon)$ is positive for $\varepsilon > 0$ and null for $\varepsilon = 0$, in keeping with the fact that $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is positive semi-definite for masonry-like materials (Lucchesi et al., 1996). $\square$

As in the case of masonry-like materials, for $\varepsilon < 1$, the derivative $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is not continuous with respect to $\mathbf{E}$ across the interfaces $\mathscr{I}_1(\varepsilon)$ and $\mathscr{I}_2(\varepsilon)$. Nevertheless, the jump of $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ at $\mathscr{I}_1(\varepsilon)$ and $\mathscr{I}_2(\varepsilon)$ satisfies the conditions given in Curnier et al. (1995), which express the absence of tangential discontinuities of the derivative of stress with respect to strain. In fact, from Eqs. (85) and (87) it can be easily shown that

$$[D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)] = -\frac{\lambda(2+\alpha)(1-\varepsilon)}{\varepsilon+(1-\varepsilon)\alpha}\nabla g_1(\mathbf{E};\varepsilon)\otimes\nabla g_1(\mathbf{E};\varepsilon) \quad \forall\ \mathbf{E}\in\mathscr{I}_1(\varepsilon) \tag{96}$$

with

$$\nabla g_1(\mathbf{E};\varepsilon) = (\varepsilon+(1-\varepsilon)\alpha)\mathbf{O}_1 + (2+\alpha)(1-\varepsilon)\mathbf{O}_2. \tag{97}$$

Moreover, in view of Eqs. (87) and (86), we have

$$[D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)] = (b(\varepsilon)-a(\varepsilon))\nabla g_2(\mathbf{E};\varepsilon)\otimes\nabla g_2(\mathbf{E};\varepsilon) \quad \forall\ \mathbf{E}\in\mathscr{I}_2(\varepsilon) \tag{98}$$

with

$$\nabla g_2(\mathbf{E};\varepsilon) = \mathbf{O}_1. \tag{99}$$

In particular, for $\mathbf{E}\in\mathscr{I}_1(\varepsilon)$ or $\mathbf{E}\in\mathscr{I}_2(\varepsilon)$, it holds that

$$[D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)][\mathbf{E}] = \mathbf{0}; \tag{100}$$

moreover, it is easy to prove that

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)[\mathbf{E}] = \widehat{\mathbf{T}}(\mathbf{E};\varepsilon) \quad \forall\ \mathbf{E}\in\mathscr{R}_i(\varepsilon), \quad i=1,2,3. \tag{101}$$

The behavior of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ as function of ε is the subject of the following statement:

Proposition 3. (*i*) *There exists a function* $f_1(\mu,\lambda,\varepsilon)$ *with*

$$\lim_{\varepsilon\to 1} f_1(\mu,\lambda,\varepsilon) = \frac{\lambda}{2+\alpha} > 0 \tag{102}$$

such that for each $\varepsilon\in(\frac{2}{3},1)$ *it holds that*

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};1)|| \leqslant (1-\varepsilon)f_1(\mu,\lambda,\varepsilon)||\mathbf{E}|| \quad \forall\ \mathbf{E}\in \mathrm{Sym}. \tag{103}$$

(*ii*) *There exists a function* $f_0(\mu,\lambda,\varepsilon)$ *with*

$$\lim_{\varepsilon\to 0} f_0(\mu,\lambda,\varepsilon) > 0 \tag{104}$$

such that for each $\varepsilon\in(0,\frac{2}{3})$ *the inequality*

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};0)|| \leqslant \varepsilon f_0(\mu,\lambda,\varepsilon)||\mathbf{E}|| \quad \forall\ \mathbf{E}\in \mathrm{Sym} \tag{105}$$

holds.

Proof. (i) If $\mathbf{E}\in\overline{\mathscr{R}_1(\varepsilon)}$, then

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};1)|| = 0. \tag{106}$$

For $\mathbf{E}\in\overline{\mathscr{R}_2(\varepsilon)}$, we have

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};1)|| = (1-\varepsilon)^2||\mathbb{D}(\varepsilon)[\mathbf{E}]||, \tag{107}$$

where

$$\mathbb{D}(\varepsilon) = \frac{\lambda(3\varepsilon-2)}{(1-\varepsilon)\alpha+\varepsilon}\mathbb{I} - \lambda\mathbf{I}\otimes\mathbf{I}. \tag{108}$$

For values of ε belonging to $(\frac{2}{3},1)$, we have $||\mathbb{D}(\varepsilon)|| = \lambda(3\varepsilon-2)/((1-\varepsilon)\alpha+\varepsilon)$ and from Eq. (107), it follows that

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};1)|| \leqslant (1-\varepsilon)^2\frac{\lambda(3\varepsilon-2)}{(1-\varepsilon)\alpha+\varepsilon}||\mathbf{E}||. \tag{109}$$

Finally, let us consider the case of $\mathbf{E}\in\overline{\mathscr{R}_3(\varepsilon)}$. It holds that for ε belonging to $[\frac{2}{3},1]$,

$$\begin{aligned}||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};1)||^2 = (1-\varepsilon)^2\Bigg\{&\left[\left(\frac{a(\varepsilon)-2\mu}{1-\varepsilon}-\lambda(1-\varepsilon)\right)^2+\lambda^2(1-\varepsilon)^2\right]e_1^2\\ &+\left[\left(\frac{b(\varepsilon)-2\mu}{1-\varepsilon}-\lambda(1-\varepsilon)\right)^2+\lambda^2(1-\varepsilon)^2\right]e_2^2\\ &-2\lambda(1-\varepsilon)\left(\frac{a(\varepsilon)-2\mu}{1-\varepsilon}+\frac{b(\varepsilon)-2\mu}{1-\varepsilon}-2\lambda(1-\varepsilon)\right)e_1e_2\Bigg\}\end{aligned} \tag{110}$$

from which it follows that

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};1)||^2 \leqslant (1-\varepsilon)^2\left\{\left(\frac{a(\varepsilon)-2\mu}{1-\varepsilon}-\lambda(1-\varepsilon)\right)^2+\lambda^2(1-\varepsilon)^2\right\}||\mathbf{E}||^2. \tag{111}$$

By comparing Eq. (111) with Eq. (109), we get Eq. (103) with

$$f_1(\mu,\lambda,\varepsilon) = \sqrt{\left(\frac{a(\varepsilon)-2\mu}{1-\varepsilon}-\lambda(1-\varepsilon)\right)^2+\lambda^2(1-\varepsilon)^2}. \tag{112}$$

(ii) Considering $\mathbf{E}\in\overline{\mathscr{R}_2(\varepsilon)}$, from Eq. (17) we have

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};0)|| = ||\mathbb{C}(\varepsilon)[\mathbf{E}]|| \leqslant ||\mathbb{C}(\varepsilon)||\,||\mathbf{E}|| = \varepsilon\sqrt{\frac{b^2(\varepsilon)}{\varepsilon^2}+4\frac{\lambda_3(\varepsilon)}{\varepsilon^2}(b(\varepsilon)+\lambda_3(\varepsilon))}||\mathbf{E}||. \tag{113}$$

For $\mathbf{E}\in\overline{\mathscr{R}_3(\varepsilon)}$, Eq. (18) implies that

$$\begin{aligned}||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};0)||^2 = \varepsilon^2\Bigg\{&\left[\left(\frac{a(\varepsilon)+\lambda_3(\varepsilon)-\varphi}{\varepsilon}\right)^2+\frac{\lambda_3^2(\varepsilon)}{\varepsilon^2}\right]e_1^2+\left[\left(\frac{b(\varepsilon)+\lambda_3(\varepsilon)}{\varepsilon}\right)^2+\frac{\lambda_3^2(\varepsilon)}{\varepsilon^2}\right]e_2^2\\ &+2\frac{\lambda_3(\varepsilon)}{\varepsilon}\left(\frac{a(\varepsilon)+b(\varepsilon)+2\lambda_3(\varepsilon)-\varphi}{\varepsilon}\right)e_1e_2\Bigg\}.\end{aligned} \tag{114}$$

As $a(\varepsilon)+b(\varepsilon)+2\lambda_3(\varepsilon)-\varphi>0$, $e_1<0$ and $e_2>0$, from Eq. (114), it follows that

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)-\widehat{\mathbf{T}}(\mathbf{E};0)|| = \varepsilon q(\mu,\lambda,\varepsilon)||\mathbf{E}||, \tag{115}$$

where

$$q(\mu,\lambda,\varepsilon) = \max\left\{\sqrt{\left(\frac{a(\varepsilon)+\lambda_3(\varepsilon)-\varphi}{\varepsilon}\right)^2+\frac{\lambda_3^2(\varepsilon)}{\varepsilon^2}},\sqrt{\left(\frac{b(\varepsilon)+\lambda_3(\varepsilon)}{\varepsilon}\right)^2+\frac{\lambda_3^2(\varepsilon)}{\varepsilon^2}}\right\}. \tag{116}$$

For $\mathbf{E} \in \overline{\mathscr{R}_1(\varepsilon)} \cap \overline{\mathscr{R}_1(0)}$, we have

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E};0)|| = 0. \tag{117}$$

Finally, let us consider the case of $\mathbf{E} \in \overline{\mathscr{R}_1(\varepsilon)} \cap \overline{\mathscr{R}_3(0)}$. For $\mathbf{E} = e_1\mathbf{O}_1 + e_2\mathbf{O}_1$, we have $e_2 = o(\varepsilon)e_1$, where $o(\varepsilon) \in [-\alpha/(2+\alpha) - \varepsilon/((1-\varepsilon)(2+\alpha)), -\alpha/(2+\alpha)]$ and thus,

$$\begin{aligned} ||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E};0)|| &= ||(2\mu - \varphi + \lambda(1 + o(\varepsilon))e_1\mathbf{O}_1 + (2\mu o(\varepsilon) + \lambda(1 + o(\varepsilon))e_2\mathbf{O}_2|| \\ &\leqslant \varepsilon\overline{y}(\mu,\lambda)||\mathbf{E}|| \end{aligned} \tag{118}$$

with

$$\overline{y}(\mu,\lambda) = \max_{t\in[0,1]} \sqrt{2\mu^2(2 + 2\alpha + \alpha^2)(1 + 2(\alpha - 1)t + (1 - 2\alpha)t^2)}. \tag{119}$$

By accounting for Eqs. (113), (115), (117) and (118), condition (105) is satisfied by

$$f_0(\mu,\lambda,\varepsilon) = \max\left\{\sqrt{\frac{b^2(\varepsilon)}{\varepsilon^2} + 4\frac{\lambda_3(\varepsilon)}{\varepsilon^2}(b(\varepsilon) + \lambda_3(\varepsilon))}, q(\mu,\lambda,\varepsilon), \overline{y}(\mu,\lambda)\right\}. \qquad \square \tag{120}$$

3. The boundary-value problem

Let $\mathscr{B}$ be a body made of a $\mathscr{M}(\varepsilon)$ material, for $\varepsilon \in [0, 1]$. More precisely, let $\mathscr{B}$ be a bounded open connected subset of the two-dimensional euclidean space, with a Lipschitz-continuous boundary $\partial\mathscr{B}$, and let $\mathscr{S}_u$ and $\mathscr{S}_f$ be two subsets of the boundary $\partial\mathscr{B}$ of $\mathscr{B}$, such that their union covers $\partial\mathscr{B}$, their interiors are disjointed and the measure of $\mathscr{S}_u$ is positive (Ciarlet, 1978).

We denote by $L^2(\mathscr{B}, \mathrm{Sym})$ the space of all symmetric tensor-valued functions $\mathbf{A}$ square integrable over $\mathscr{B}$, with the norm

$$||\mathbf{A}||_{L^2(\mathscr{B},\mathrm{Sym})} = \left(\int_{\mathscr{B}} ||\mathbf{A}(x)||^2\,\mathrm{d}x\right)^{1/2}, \tag{121}$$

where $||\ ||$ is the norm of Lin, and by $H^1(\mathscr{B}, \mathscr{V})$, the space of all vector-valued functions $\mathbf{v}$ belonging to $L^2(\mathscr{B}, \mathscr{V})$ such that $\nabla\mathbf{v} \in L^2(\mathscr{B}, \mathrm{Sym})$, with the norm

$$||\mathbf{v}||_{H^1(\mathscr{B},\mathscr{V})} = \left(||\mathbf{v}||^2_{L^2(\mathscr{B},\mathscr{V})} + ||\nabla\mathbf{v}||^2_{L^2(\mathscr{B},\mathrm{Sym})}\right)^{1/2}. \tag{122}$$

Moreover, let us consider the sets

$$\mathscr{Y} = \{\mathbf{A} \in L^2(\mathscr{B}, \mathrm{Sym}) \mid \operatorname{div}\mathbf{A} \in L^2(\mathscr{B}, \mathscr{V})\}, \tag{123}$$

$$\mathscr{U} = \{\mathbf{v} \in H^1(\mathscr{B}, \mathscr{V}) \mid \mathbf{v} = \mathbf{0} \text{ on } \mathscr{S}_u\}. \tag{124}$$

Let $\mathbf{b} \in L^2(\mathscr{B}, \mathscr{V})$ and $\mathbf{s}_0 \in L^2(\mathscr{S}_f, \mathscr{V})$ be two given functions. A load $(\mathbf{b}, \mathbf{s}_0)$ is admissible if the corresponding boundary-value problem has a solution, i.e., if there exists a triple $[\mathbf{u}_\varepsilon, \mathbf{E}_\varepsilon, \mathbf{T}_\varepsilon]$ constituted by a stress field $\mathbf{T}_\varepsilon \in \mathscr{Y}$, a strain field $\mathbf{E}_\varepsilon \in L^2(\mathscr{B}, \mathrm{Sym})$ and a displacement field $\mathbf{u}_\varepsilon \in \mathscr{U}$, such that

$$\mathbf{E}_\varepsilon = \tfrac{1}{2}(\nabla\mathbf{u}_\varepsilon + \nabla\mathbf{u}_\varepsilon^{\mathrm{T}}), \tag{125}$$

$$\mathbf{T}_\varepsilon = \widehat{\mathbf{T}}(\mathbf{E}_\varepsilon;\varepsilon), \tag{126}$$

$$\mathbf{T}_\varepsilon\mathbf{n} = \mathbf{s}_0, \text{ on } \mathscr{S}_f, \tag{127}$$

$$\operatorname{div}\mathbf{T}_\varepsilon + \mathbf{b} = \mathbf{0}, \tag{128}$$

where $\mathbf{n}$ is the outward unit normal to $\mathscr{S}_f$. It is easy to prove that, if $[\mathbf{u}_\varepsilon^{(1)}, \mathbf{E}_\varepsilon^{(1)}, \mathbf{T}_\varepsilon^{(1)}]$ and $[\mathbf{u}_\varepsilon^{(2)}, \mathbf{E}_\varepsilon^{(2)}, \mathbf{T}_\varepsilon^{(2)}]$ are two solutions to Eqs. (125)–(128) for $\varepsilon > 0$, then the two coincide in $\mathscr{B}$. The triple $[\overline{\mathbf{u}}_\varepsilon, \overline{\mathbf{E}}_\varepsilon, \overline{\mathbf{T}}_\varepsilon]$, with $\overline{\mathbf{u}}_\varepsilon = \mathbf{u}_\varepsilon^{(1)} - \mathbf{u}_\varepsilon^{(2)}$, $\overline{\mathbf{E}}_\varepsilon = \mathbf{E}_\varepsilon^{(1)} - \mathbf{E}_\varepsilon^{(2)}$, $\overline{\mathbf{T}}_\varepsilon = \mathbf{T}_\varepsilon^{(1)} - \mathbf{T}_\varepsilon^{(2)}$, satisfies Eq. (125) and $\overline{\mathbf{u}}_\varepsilon = \mathbf{0}$ on $\mathscr{S}_u$. Moreover, it satisfies Eqs. (127) and (128) with $\mathbf{b} = \mathbf{0}$ and $\mathbf{s}_0 = \mathbf{0}$. Thus, a simple application of the principle of virtual work proves that

$$\int_{\mathscr{B}} \overline{\mathbf{T}}_\varepsilon \cdot \overline{\mathbf{E}}_\varepsilon \,\mathrm{d}x = 0. \tag{129}$$

Now, by virtue of the strong monotonicity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$, from Eq. (129) we may deduce that $\overline{\mathbf{T}}_\varepsilon \cdot \overline{\mathbf{E}}_\varepsilon = 0$ in $\mathscr{B}$ and therefore, once more using Eq. (48), can conclude that $\mathbf{E}_\varepsilon^{(1)} = \mathbf{E}_\varepsilon^{(2)}$ and then $\mathbf{T}_\varepsilon^{(1)} = \mathbf{T}_\varepsilon^{(2)}$. Considering that $\nabla\mathbf{u}_\varepsilon^{(1)} = \nabla\mathbf{u}_\varepsilon^{(2)}$ and $\mathbf{u}_\varepsilon^{(1)} = \mathbf{u}_\varepsilon^{(2)}$ on $\mathscr{S}_u$, we get $\mathbf{u}_\varepsilon^{(1)} = \mathbf{u}_\varepsilon^{(2)}$ in $\mathscr{B}$. On the contrary, if $\varepsilon = 0$, Eq. (129) allows us to conclude only that $\mathbf{T}_0^{(1)} = \mathbf{T}_0^{(2)}$ (Lucchesi et al., 1996).

Now, let us consider the equilibrium problem (125)–(128) and, for $\varepsilon \in (0,1)$, let $[\mathbf{u}_\varepsilon, \mathbf{E}_\varepsilon, \mathbf{T}_\varepsilon]$ be its solution defined on $\mathscr{B}$ with values in $\mathscr{V} \times \mathrm{Sym} \times \mathrm{Sym}$. We denote by $[\mathbf{u}_1, \mathbf{E}_1, \mathbf{T}_1]$ the solution to the equilibrium problem corresponding to a linear elastic material $\mathscr{M}(1)$, and by $[\mathbf{u}_0, \mathbf{E}_0, \mathbf{T}_0]$ a solution to the equilibrium problem for a masonry-like material $\mathscr{M}(0)$. No assumptions are made on the uniqueness of displacement $\mathbf{u}_0$ or strain $\mathbf{E}_0$. We analyze the behavior of $[\mathbf{u}_\varepsilon, \mathbf{E}_\varepsilon, \mathbf{T}_\varepsilon]$ for ε tending towards 1 and 0.

Proposition 4. *The following results hold:*

$$\lim_{\varepsilon\to 1} \mathbf{u}_\varepsilon = \mathbf{u}_1 \quad \textit{in } H^1(\mathscr{B}, \mathscr{V}), \tag{130}$$

$$\lim_{\varepsilon\to 1} \mathbf{E}_\varepsilon = \mathbf{E}_1, \quad \lim_{\varepsilon\to 1} \mathbf{T}_\varepsilon = \mathbf{T}_1 \quad \textit{in } L^2(\mathscr{B}, \mathrm{Sym}); \tag{131}$$

$$\lim_{\varepsilon\to 0} \mathbf{T}_\varepsilon = \mathbf{T}_0 \quad \textit{in } L^2(\mathscr{B}, \mathrm{Sym}). \tag{132}$$

Proof. (i) In view of the principle of virtual work, it holds that for each $\varepsilon \in (0,1]$,

$$\int_{\mathscr{B}} (\mathbf{T}_\varepsilon(x) - \mathbf{T}_1(x)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)) \,\mathrm{d}x = 0. \tag{133}$$

The following relations

$$\begin{aligned}
(\mathbf{T}_\varepsilon(x) - \mathbf{T}_1(x)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)) &= (\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_1(x);1)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)) \\
&= (\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_1(x);\varepsilon)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)) \\
&\quad + (\widehat{\mathbf{T}}(\mathbf{E}_1(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_1(x);1)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)) \\
&\geqslant \kappa(\varepsilon)||\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)||^2 - ||\widehat{\mathbf{T}}(\mathbf{E}_1(x);\varepsilon) \\
&\quad - \widehat{\mathbf{T}}(\mathbf{E}_1(x);1)||\,||\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)||
\end{aligned} \tag{134}$$

also hold, for which we have used the strong monotonicity of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ given in Eq. (48) and the Schwarz inequality. By integrating Eq. (134) and accounting for Eq. (133), we show that for each $\varepsilon \in (0,1]$,

$$\begin{aligned}\int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\|^2 \,\mathrm{d}x &\leqslant \frac{1}{\kappa(\varepsilon)} \int_{\mathscr{B}} \|\widehat{\mathbf{T}}(\mathbf{E}_1(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_1(x);1)\| \, \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\| \,\mathrm{d}x \\ &\leqslant \frac{1-\varepsilon}{\kappa(\varepsilon)} f_1(\mu,\lambda,\varepsilon) \int_{\mathscr{B}} \|\mathbf{E}_1(x)\| \, \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\| \,\mathrm{d}x \\ &\leqslant \frac{1-\varepsilon}{\kappa(\varepsilon)} f_1(\mu,\lambda,\varepsilon) \left(\int_{\mathscr{B}} \|\mathbf{E}_1(x)\|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\|^2 \,\mathrm{d}x \right)^{1/2}, \end{aligned} \tag{135}$$

where the second inequality follows from Eq. (103) and the third from the Schwarz inequality. From Eq. (135), keeping in mind that Eq. (102) and $\kappa(1) = 2\mu$ hold, we obtain

$$\lim_{\varepsilon \to 1} \int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\|^2 \,\mathrm{d}x = 0, \tag{136}$$

which expresses the convergence of $\mathbf{E}_\varepsilon$ on $\mathbf{E}_1$ with respect to the L^2 norm, when ε tends towards to 1.

By virtue of the strict positiveness of the measure of $\mathscr{S}_u$, over the space $\mathscr{U}$ the norm $\|\mathbf{v}\|_{H^1(\mathscr{B},\mathscr{V})}$ is equivalent to the norm $|\mathbf{v}| = \|\frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}})\|_{L^2(\mathscr{B},\mathrm{Sym})}$ (Ciarlet, 1978) , thus the convergence

$$\lim_{\varepsilon \to 1} \|\mathbf{E}_\varepsilon - \mathbf{E}_1\|_{L^2(\mathscr{B},\mathrm{Sym})} = 0 \tag{137}$$

implies the convergence

$$\lim_{\varepsilon \to 1} \|\mathbf{u}_\varepsilon - \mathbf{u}_1\|_{H^1(\mathscr{B},\mathscr{V})} = 0. \tag{138}$$

As far as the convergence of $\mathbf{T}_\varepsilon$ is concerned, let us consider the following inequalities:

$$\begin{aligned}\|\mathbf{T}_\varepsilon(x) - \mathbf{T}_1(x)\| &\leqslant \|\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);1)\| + \|\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);1) - \widehat{\mathbf{T}}(\mathbf{E}_1(x);1)\| \\ &\leqslant (1-\varepsilon) f_1(\mu,\lambda,\varepsilon)\|\mathbf{E}_\varepsilon(x)\| + 2\mu\|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\|. \end{aligned} \tag{139}$$

By taking the square of Eq. (139) and integrating over $\mathscr{B}$, we get

$$\begin{aligned}\int_{\mathscr{B}} \|\mathbf{T}_\varepsilon(x) - \mathbf{T}_1(x)\|^2 \,\mathrm{d}x &\leqslant (1-\varepsilon)^2 f_1(\mu,\lambda,\varepsilon)^2 \int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x)\|^2 \,\mathrm{d}x + 4\mu^2 \int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\|^2 \,\mathrm{d}x \\ &\quad + 4\mu(1-\varepsilon) f_1(\mu,\lambda,\varepsilon) \left(\int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x)\|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\mathscr{B}} \|\mathbf{E}_\varepsilon(x) - \mathbf{E}_1(x)\|^2 \,\mathrm{d}x \right)^{1/2} \end{aligned} \tag{140}$$

from which

$$\lim_{\varepsilon \to 1} \int_{\mathscr{B}} \|\mathbf{T}_\varepsilon(x) - \mathbf{T}_1(x)\|^2 \,\mathrm{d}x = 0 \tag{141}$$

follows.

(ii) The principle of virtual work implies that for every $\varepsilon \in (0, 1]$,

$$\int_{\mathscr{B}} (\mathbf{T}_\varepsilon(x) - \mathbf{T}_0(x)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)) \,\mathrm{d}x = 0. \tag{142}$$

By proceeding as in the previous part (i) and accounting for Eqs. (48) and (105), we get

$$\begin{aligned}\int_{\mathscr{B}} ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||^2\,\mathrm{d}x &\leqslant \frac{1}{\kappa(\varepsilon)} \int_{\mathscr{B}} ||\widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);0)||\ ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||\,\mathrm{d}x \\ &\leqslant \frac{\varepsilon}{\kappa(\varepsilon)} f_0(\mu,\lambda,\varepsilon) \int_{\mathscr{B}} ||\mathbf{E}_0(x)||\ ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||\,\mathrm{d}x \\ &\leqslant \frac{\varepsilon}{\kappa(\varepsilon)} f_0(\mu,\lambda,\varepsilon) \left(\int_{\mathscr{B}} ||\mathbf{E}_0(x)||^2\,\mathrm{d}x \right)^{1/2} \left(\int_{\mathscr{B}} ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||^2\,\mathrm{d}x \right)^{1/2} \end{aligned} \tag{143}$$

for each $\varepsilon \in (0, 1]$, from which we obtain

$$\int_{\mathscr{B}} ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||^2\,\mathrm{d}x \leqslant \gamma \quad \forall\ \varepsilon \in (0, 1]; \tag{144}$$

in view of Eqs. (37) and (53), γ has the expression

$$\gamma = \left(\max_{\varepsilon} \frac{(1-\varepsilon)\alpha + \varepsilon}{\mu[2 + \alpha(1-\varepsilon)(5-3\varepsilon)]} f_0(\mu,\lambda,\varepsilon) \right)^2 \int_{\mathscr{B}} ||\mathbf{E}_0(x)||^2\,\mathrm{d}x. \tag{145}$$

Taking Eq. (49) into account, for each $\varepsilon \in (0, 1]$, we have

$$\begin{aligned}(\mathbf{T}_\varepsilon(x) - \mathbf{T}_0(x)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)) &= (\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);0)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)) \\ &= (\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)) \\ &\quad + (\widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);0)) \cdot (\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)) \\ &\geqslant \frac{1}{2(\mu+\lambda)} ||\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon)||^2 \\ &\quad - ||\widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);0)||\ ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||. \end{aligned} \tag{146}$$

By integrating Eq. (146) and recalling Eq. (105), we arrive at

$$\int_{\mathscr{B}} ||\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon)||^2\,\mathrm{d}x \leqslant 2(\mu+\lambda)\varepsilon f_0(\mu,\lambda,\varepsilon) \left(\int_{\mathscr{B}} ||\mathbf{E}_\varepsilon(x) - \mathbf{E}_0(x)||^2\,\mathrm{d}x \right)^{1/2} \tag{147}$$

for each $\varepsilon \in (0, 1]$. By accounting for Eq. (144), and considering the limit of Eq. (147), for ε tending towards 0, we get

$$\lim_{\varepsilon \to 0} \int_{\mathscr{B}} ||\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon)||^2\,\mathrm{d}x = 0. \tag{148}$$

Finally, by integrating the relation

$$||\mathbf{T}_\varepsilon(x) - \mathbf{T}_0(x)||^2 \leqslant (||\widehat{\mathbf{T}}(\mathbf{E}_\varepsilon(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon)|| + ||\widehat{\mathbf{T}}(\mathbf{E}_0(x);\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E}_0(x);0)||)^2, \tag{149}$$

over $\mathscr{B}$ and considering Eqs. (105) and (148), we obtain Eq. (132).

We note that in order to obtain Eq. (132), no assumptions need be made on the uniqueness of the displacement or strain fields corresponding to the masonry material. □

4. Conclusions

A class of non-linear hyperelastic materials $\mathscr{M}(\varepsilon)$ dependent on parameter ε has been introduced. For $\varepsilon = 0$, $\mathscr{M}(0)$ is a masonry-like material, and for $\varepsilon = 1$, material $\mathscr{M}(1)$ is linear elastic. The main properties of stress function $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ have been proved, in particular it has been shown that for $\epsilon > 0$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is strongly monotone with respect to $\mathbf{E}$. This allows proving that the solution to the equilibrium problem of solids made of a material having constitutive equation $\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is unique in terms of displacement, strain and stress; on the contrary for masonry-like materials, the solution is unique only in terms of stress.

Results given in Section 3, dealing with the behavior, for ε going to 0 or 1, of the solution $[\mathbf{u}_\varepsilon, \mathbf{E}_\varepsilon, \mathbf{T}_\varepsilon]$ to the equilibrium problem of a solid made of a material $\mathscr{M}(\varepsilon)$, are general. They do not depend on the particular stress function $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ chosen, and hold for all functions $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ strongly monotone for $\varepsilon > 0$ and monotone for $\varepsilon = 0$.

Appendix A

Let us now consider the case $\lambda = 0$. For

$$g_1(\mathbf{E}) = e_2, \tag{A.1}$$

$$g_2(\mathbf{E}) = e_1, \tag{A.2}$$

let us define the hypersurfaces

$$\mathscr{I}_1 = \{\mathbf{E} \in \mathrm{Sym} \mid g_1(\mathbf{E}) = 0\}, \tag{A.3}$$

$$\mathscr{I}_2 = \{\mathbf{E} \in \mathrm{Sym} \mid g_2(\mathbf{E}) = 0\}, \tag{A.4}$$

and the open cones of Sym

$$\mathscr{R}_1 = \{\mathbf{E} \in \mathrm{Sym} \mid g_1(\mathbf{E}) < 0\}, \tag{A.5}$$

$$\mathscr{R}_2 = \{\mathbf{E} \in \mathrm{Sym} \mid g_2(\mathbf{E}) > 0\}, \tag{A.6}$$

$$\mathscr{R}_3 = \{\mathbf{E} \in \mathrm{Sym} \mid g_1(\mathbf{E}) > 0, g_2(\mathbf{E}) < 0\}. \tag{A.7}$$

$\mathscr{R}_1$ and $\mathscr{R}_2$, coinciding with the cones of the negative definite and positive definite tensors, respectively, are convex, while $\mathscr{R}_3$ is not. For $\varepsilon \in [0,1]$, let us consider the following function $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$, dependent upon $\mathbf{E} \in \mathrm{Sym}$, with values in Sym

$$\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = 2\mu\mathbf{E}, \quad \mathbf{E} \in \overline{\mathscr{R}_1}, \tag{A.8}$$

$$\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = 2\mu\varepsilon\mathbf{E}, \quad \mathbf{E} \in \overline{\mathscr{R}_2}, \tag{A.9}$$

$$\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = 2\mu e_1\mathbf{O}_1 + 2\mu\varepsilon e_2\mathbf{O}_2, \quad \mathbf{E} \in \overline{\mathscr{R}_3}. \tag{A.10}$$

The following results hold:

(i) For every $\varepsilon \in (0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is strongly monotone

$$(\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{U};\varepsilon)) \cdot (\mathbf{E} - \mathbf{U}) \geqslant 2\mu\varepsilon||\mathbf{E} - \mathbf{U}||^2 \quad \forall\ \mathbf{E}, \mathbf{U} \in \mathrm{Sym}. \tag{A.11}$$

(ii) For every $\varepsilon \in [0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is Lipschitz continuous

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{U};\varepsilon)|| \leqslant 2\mu||\mathbf{E} - \mathbf{U}|| \quad \forall\ \mathbf{E}, \mathbf{U} \in \mathrm{Sym}. \tag{A.12}$$

(iii) For every $\varepsilon \in [0,1]$, $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is monotone

$$(\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{U};\varepsilon)) \cdot (\mathbf{E} - \mathbf{U}) \geqslant \frac{1}{2\mu}||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{U};\varepsilon)||^2 \quad \forall\ \mathbf{E}, \mathbf{U} \in \mathrm{Sym}. \tag{A.13}$$

The proof of (i)–(iii) is similar to that given, in Proposition 1 for $\lambda > 0$ and is omitted here. As far as the dependence of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ on ε is concerned, it is easy to verify that for each $\mathbf{E} \in \mathrm{Sym}$

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E};1)|| \leqslant 2\mu(1-\varepsilon)||\mathbf{E}||, \tag{A.14}$$

$$||\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) - \widehat{\mathbf{T}}(\mathbf{E};0)|| \leqslant 2\mu\varepsilon||\mathbf{E}||. \tag{A.15}$$

$\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is differentiable in every region $\mathscr{R}_i$. In fact, the derivative $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ of $\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ with respect to $\mathbf{E}$ is

$$\text{for } \mathbf{E} \in \mathscr{R}_1, \qquad D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = 2\mu\mathbb{I}, \tag{A.16}$$

$$\text{for } \mathbf{E} \in \mathscr{R}_2, \qquad D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = 2\mu\varepsilon\mathbb{I}, \tag{A.17}$$

$$\text{for } \mathbf{E} \in \mathscr{R}_3, \qquad D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon) = 2\mu\mathbf{O}_1 \otimes \mathbf{O}_1 + 2\mu\varepsilon\mathbf{O}_2 \otimes \mathbf{O}_2 + \frac{2\mu\varepsilon e_2 - 2\mu e_1}{e_2 - e_1}\mathbf{O}_3 \otimes \mathbf{O}_3. \tag{A.18}$$

If $\varepsilon < 1$, the derivative $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is not continuous with respect to $\mathbf{E}$ across interfaces $\mathscr{I}_1$ and $\mathscr{I}_2$, but the jumps of $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ in correspondence to $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively, satisfy the conditions

$$[D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)] = 2\mu(1-\varepsilon)\mathbf{O}_2 \otimes \mathbf{O}_2 \quad \forall\ \mathbf{E} \in \mathscr{I}_1, \tag{A.19}$$

$$[D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)] = 2\mu(1-\varepsilon)\mathbf{O}_1 \otimes \mathbf{O}_1 \quad \forall\ \mathbf{E} \in \mathscr{I}_2. \tag{A.20}$$

For masonry materials, $D_E\widehat{\mathbf{T}}(\mathbf{E};0)$ is positive semi-definite in each region $\mathscr{R}^i$ and $D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)$ is positive definite in $\mathscr{R}_i$ for $\varepsilon > 0$; in fact, for every $\mathbf{E}$ belonging to $\mathscr{R}_i$

$$D_E\widehat{\mathbf{T}}(\mathbf{E};\varepsilon)[\mathbf{H}] \cdot \mathbf{H} \geqslant 2\mu\varepsilon||\mathbf{H}||^2 \quad \forall\ \mathbf{H} \in \mathrm{Sym}. \tag{A.21}$$

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